16.5 Curl and Divergence

Def:- The "del" operator is defined as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

We have already seen how ∇ acts on scalar fields $f(x,y,z)$. It produces the gradient of f or ∇f.

$$\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \langle f_x, f_y, f_z \rangle$$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Example: Find gradient of $\boxed{f(x,y,z) = x\sin y + y\cos z}$

$$\underline{\nabla f = \langle \sin y,\ x\cos y + \cos z,\ -y\sin z \rangle.}$$

In this section, we use "∇" to define two other important differential operators acting on vector fields $\vec{F} = \langle P, Q, R \rangle$.

$$\boxed{\begin{array}{l} \text{Curl } \vec{F} = \nabla \times \vec{F} \quad \text{(cross product)} \\ \hline \text{div } \vec{F} = \nabla \cdot \vec{F} \quad \text{(dot product)} \end{array}}$$

<u>Def.</u>- $\vec{F}$ is a vector field defined on $D \subseteq \mathbb{R}^3$.

All Components of $\vec{F}$ have partial derivatives then,

$$\text{Curl } \vec{F} = \text{Curl } \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\overset{\text{formal Cross product}}{=} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \nabla \times \vec{F}$$

So $\nabla \times = \text{Curl}$.

<u>Example</u>:- Find Curl $\vec{F}$ for (Exercise 15 Sect. 16.5)

$$\vec{F}(x,y,z) = \langle z\cos y, xz\sin y, x\cos y \rangle$$

<u>Answ</u>:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z\cos y & xz\sin y & x\cos y \end{vmatrix} =$$

$$= \langle -x\sin y - x\sin y, \cos y - \cos y, z\sin y + z\sin y \rangle$$

$$= \langle -2x\sin y, 0, 2z\sin y \rangle$$

Example: Find Curl $\vec{F}$ for

$$\vec{F} = \langle \sin y, x\cos y + \cos z, -y\sin z \rangle$$

answ:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin y & x\cos y + \cos z & -y\sin z \end{vmatrix}$$

$$= \langle -\sin z + \sin z,\ 0 - 0,\ \cos y - \cos y \rangle = \langle 0,0,0 \rangle = \vec{0}.$$

Notice that $\vec{F}$ is conservative because

$$\vec{F} = \underline{\langle \sin y, x\cos y + \cos z, -y\sin z \rangle} = \nabla f(x,y,z)$$

where

$$\boxed{f(x,y,z) = x\sin y + y\cos z}$$

So $\nabla \times (\nabla f) = 0.$

This is a general result.

<u>Thm.</u> If f is defined in $D \subseteq \mathbb{R}^3$ and it has Continuous second order partial derivatives then

$$\boxed{\nabla \times (\nabla f) = \vec{0}.}$$

<u>Proof:-</u>

$$\nabla \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left\langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right\rangle$$

$$= \langle 0, 0, 0 \rangle = \vec{0}.$$

Using Clairaut's thm.

<u>Corollary:-</u> If $\vec{F}$ is a conservative vector field then $\nabla \times \vec{F} = \vec{0}$.

So, if $\nabla \times \vec{F} \neq \vec{0}$ then $\vec{F}$ is not conservative.

Is the opposite true?

Thm. If $\vec{F}$ is a vector field defined on a simply connected domain $D \subseteq \mathbb{R}^3$, such that

$$\nabla \times \vec{F} = 0$$

then $\vec{F}$ is conservative.

Proof. (Requires Stoke's thm).

Finding $f(x,y,z)$ (potential function) of a conservative vector field

Example: $\vec{F} = \langle \sin y, x\cos y + \cos z, -y\sin z \rangle$

$$f_x = \sin y, \quad f_y = x\cos y + x\cos z, \quad f_z = -y\sin z$$

Then,

$$\int dx : f(x,y) = x\sin y + C(y,z)$$

$$\Rightarrow f_y(x,y) = x\cos y + \frac{\partial C}{\partial y} = x\cos y + \cos z$$

$$\overset{\int dy}{\Longrightarrow} \boxed{f(x,y) = x\sin y + y\cos z + d(z)}$$

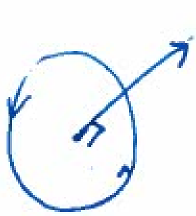
$$\Rightarrow f_z(x,y) = -y\sin z + d'(z) = -y\sin z$$

$$\Rightarrow d'(z) = 0 \Rightarrow d(z) = K$$

Finally, $\boxed{f(x,y,z) = x\sin y + y\cos z + K.}$

Physical Interpretation of $\nabla \times F$.

$\vec{F}$: Velocity of fluid particles.



$\nabla \times \vec{F}$ near
- Particles rotate about $\nabla \times \vec{F}$ axis
- length of $\nabla \times \vec{F}$ is speed of rotation.

Example: tiny paddle wheel motion

If $\nabla \times \vec{F} = \vec{0}$ then $\vec{F}$ is called irrotational.

Def. – $\vec{F} = \langle P, Q, R \rangle$ is a vector field defined on $D \subseteq \mathbb{R}^3$

– Partial derivatives of P, Q, R exists in D.

Then

$$\boxed{\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

Notice,

$$\nabla \cdot \vec{F} \overset{\text{Formal Notation}}{=} \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \underset{\text{(Dot Product)}}{\cdot} \langle P, Q, R \rangle$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Example: (#7 Sect. 16.5)

$\nabla\cdot\vec{F}$ where $\vec{F} = \langle e^x \sin y, e^y \sin z, e^z \sin x \rangle$

answ:

$$\nabla\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\ \frac{\partial R}{\partial z} = e^x \sin y + e^y \sin z + e^z \sin x$$

Example:

$\nabla\cdot\vec{G}$, where $\vec{G} = \langle -2x\sin y, 0, 2z\sin y \rangle$

answ:

$$\boxed{\nabla\cdot G = -2\sin y + 2\sin y = 0 \ !}$$

Notice that $\vec{G} = \nabla\times\vec{F}$, where $\vec{F} = \langle z\cos y, xz\sin y, x\cos y \rangle$

Then

$$\nabla\cdot(\nabla\times\vec{F}) = 0.$$

This is a particular case of a general result

Thm. $\vec{F} = \langle P, Q, R \rangle$ Vector field defined on $D \subseteq \mathbb{R}^3$

- P, Q, and R have continuous 2^{nd} order PDs.

then

$$\boxed{\operatorname{div}(\operatorname{curl}\vec{F}) = \nabla\cdot(\nabla\times\vec{F}) = 0.}$$

Proof:- Straightforward using Clairaut's thm.

If $\vec{F}$ is such that

$$\nabla \cdot \vec{F} = 0$$

then $\vec{F}$ is called incompressible.

__Def.__ If f defined in $D \subseteq \mathbb{R}^3$ has 2^{nd} order partial derivatives, then we define the Laplacian operator acting on f as

$$\boxed{\begin{aligned}&\nabla \cdot (\nabla f) = \nabla \cdot \left(\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle\right)\\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.\end{aligned}}$$

We also use the notation:

$$\boxed{\nabla^2 f = \nabla \cdot (\nabla f)}$$

The equation

$$\boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0}$$

or

$$\boxed{\nabla^2 f = 0}$$

is called the Laplace equation.